Relativistic models for pulsar glitches with realistic equations of state

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in collaboration with

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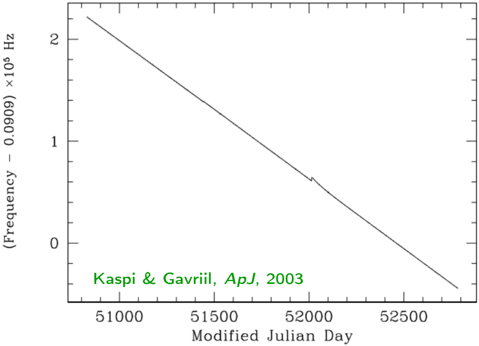
1. Introduction
   - Observations
   - Vortex-mediated glitch theory
2. Numerical models for pulsar glitches
   - Realistic equilibrium configurations
   - Bulk model for pulsar glitches
   - Towards a more realistic model
3. Conclusion

The glitch phenomenon



Kaspi & Gavriil, *ApJ*, 2003

Observational features

Espinoza et al., *MNRAS*, 2011

- **amplitude:**

$$\Delta\Omega/\Omega \sim 10^{-9} - 10^{-5}$$

- short **rise time:**

$$\boxed{\tau_r < 30 \text{ s}} \quad \dashleftarrow \text{Vela}$$

- exponential **relaxation** on several days or months.

Models for pulsar glitches

Haskell & Melatos, *IJMPD*, 2015

- Rearrangement of the moment of inertia $\dashrightarrow$ crust quakes,
- Angular momentum transfer between *two* fluids $\dashrightarrow$ **superfluidity**.

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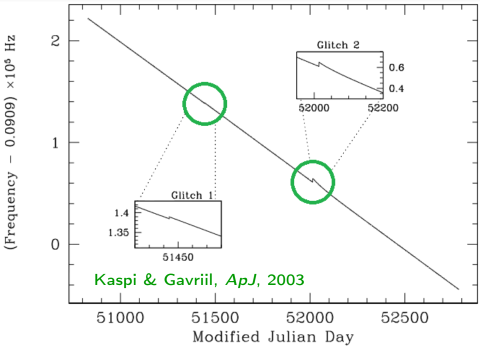

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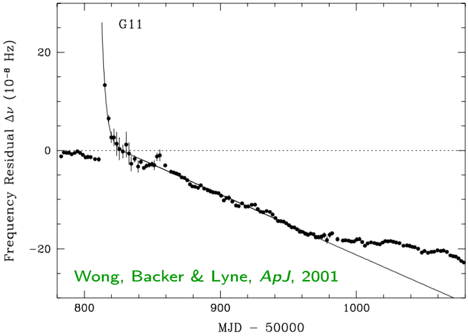

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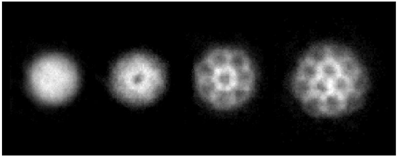
Superfluidity in neutron stars

Superfluid properties:

- **zero** viscosity,
- **infinite** thermal conductivity,
- angular momentum quantized into **vortex lines**.



Madison et al., *PRL*, 2000

Theoretical predictions for NSs

$$\boxed{T_c \simeq 10^9 - 10^{10} \text{ K}}$$

⇢ superfluid neutrons in the core & in the inner crust of NSs.

Observational evidence

- **Long relaxation time scales in pulsar glitches**,
- Fast cooling in Cassiopeia A,
- QPOs from SGRs, ...

Vortex-mediated glitch theory

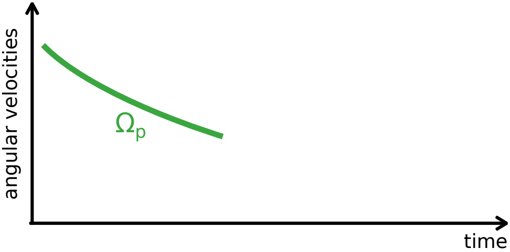
Anderson & Itoh, *Nature*, 1975

Two-fluid model

Baym et al., *Nature*, 1969

- Charged particles:

  $\Omega_p = \Omega \leftrightarrow$ pulsar



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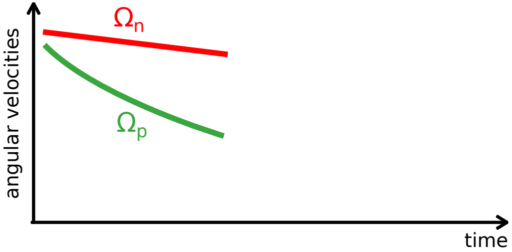
  $\Omega_n \gtrsim \Omega_p$



Key assumption: the vortices can **pin** to the crust and/or to flux tubes.

Vortex-mediated glitch theory

Anderson & Itoh, *Nature*, 1975

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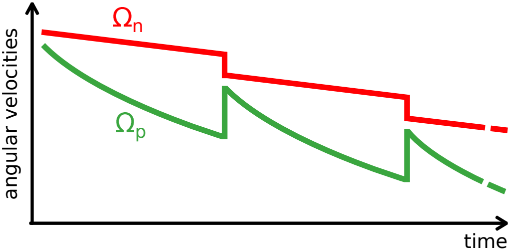
- Superfluid neutrons:

  $\Omega_{\mathrm{n}} \gtrsim \Omega_{\mathrm{p}}$



Once a critical lag $\Omega_{\mathrm{n}} - \Omega_{\mathrm{p}}$ is reached:

some vortices get **unpinned** and are allowed to move **radially**

⇢ angular momentum transfer between the fluids

Purposes of this work

Objectives:

- Build a *simple* numerical model for pulsar glitches, including **general relativity** and **realistic** equations of state,
- Get some **constraints** on the interior of neutron stars.

Fundamental hypothesis (to be verified):

hydrodynamical time ~ 0.1 ms $\ll$ **rise time** (dissipation)

⇢ the glitch event is a **quasi-stationary** process and can thus be modelled through a series of **equilibrium** configurations.

Assumptions & Ingredients

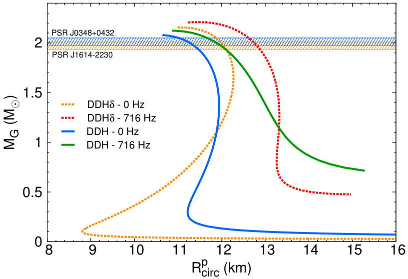
Prix et al., *PRD*, 2005 & Sourie et al., *PRD*, 2016

Equilibrium configurations:

- isolated star,
- $T = 0$,
- no magnetic field,
- dissipative effects are **neglected**,
- **uniform** composition $\rightarrow p, e^-, n$,
- asymptotically flat, **stationary**, **axisymmetric** & **circular** metric,
- **rigid-body** rotation: Ω_n, Ω_p.

Equations of state:

Density-dependent RMF models (DDH & DDHδ)



Fluid couplings

Moments of inertia:

$$\mathrm{d}J_X = I_{XX}\ \mathrm{d}\Omega_X + I_{XY}\ \mathrm{d}\Omega_Y \qquad X, Y \in \{\mathrm{n}, \mathrm{p}\}$$

$$\bar{I}_X = I_{XX} + I_{XY} \qquad \bar{I} = \bar{I}_\mathrm{n} + \bar{I}_\mathrm{p}$$

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The fluids are mainly coupled through two *non-dissipative* mechanisms:

- entrainment effect

  due to the strong interactions between nucleons *in the core*:

  $$\boxed{p^\alpha_X = \mathcal{K}^{XX} n_X u^\alpha_X + \mathcal{K}^{XY} n_Y u^\alpha_Y}$$

  Andreev & Bashkin, *SJETP*, 1976

- relativistic frame-dragging effect

  associated with the rotation of the fluids Ω_n and Ω_p:

  $$\boxed{g_{t\varphi} \neq 0}$$

  Carter, *Annals of Physics*, 1975

Entrainment VS frame-dragging

Coupling terms:

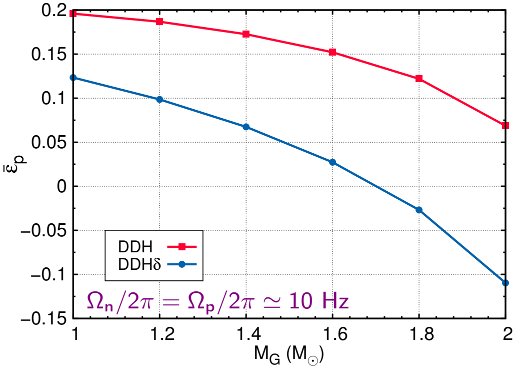
$$\bar{\varepsilon}_X = I_{XY} / \bar{I}_X$$

Total coupling, *at low* Ω:

$$\boxed{\bar{\varepsilon}_\mathrm{p} = \frac{\varepsilon_\mathrm{p}^{entr} + \varepsilon_{\mathrm{n}\to\mathrm{p}}^{LT}}{1 + \varepsilon_{\mathrm{p}\to\mathrm{p}}^{LT} + \varepsilon_{\mathrm{n}\to\mathrm{p}}^{LT}}}$$

Remarks:

- $\varepsilon^{entr} > 0$ in the core,
- $\varepsilon^{LT} < 0$.



NB: $\bar{\varepsilon}_\mathrm{n} \simeq \bar{I}_\mathrm{p}/\bar{I}_\mathrm{n} \times \bar{\varepsilon}_\mathrm{p} \simeq 0.05 \times \bar{\varepsilon}_\mathrm{p}$

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Angular momentum transfer

Langlois et al., *MNRAS*, 1998 & Sidery et al., *MNRAS*, 2010

$\Omega_\mathrm{n} - \Omega_\mathrm{p} = \delta\Omega_0 \Rightarrow$ angular momentum transfer through **mutual friction**.

Assuming *straight vortices*, the **mutual friction moment** reads

$$\Gamma_\mathrm{int} = -\overline{\mathcal{B}} \int \Gamma_\mathrm{n} n_\mathrm{n} \varpi_\mathrm{n} h_\perp^2 \, \mathrm{d}\Sigma \times (\Omega_\mathrm{n} - \Omega_\mathrm{p}) = -2\overline{\mathcal{B}}\zeta \bar{I}_\mathrm{n} \Omega_\mathrm{n} \times \delta\Omega$$

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mutual friction parameter

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mutual friction parameter

$$\boxed{\bar{\mathcal{B}} = \frac{\mathcal{R}}{1+\mathcal{R}^2}}$$

Drag-to-lift ratio $\mathcal{R}$

- *crust:* $10^{-3} \lesssim \mathcal{R} \lesssim 10^{-1}$,
- *core:* $\mathcal{R} \lesssim 10^{-3}$.

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$$\boxed{\overline{\mathcal{B}} = \frac{\mathcal{R}}{1+\mathcal{R}^2}} \leqslant \frac{1}{2}$$

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Spin-up time scale

Evolution equations:

$$\begin{cases} \dot{j}_{\rm n} & = & + \, \Gamma_{\rm int}, \\ \dot{j}_{\rm p} & = & - \, \Gamma_{\rm int}. \end{cases}$$

$$\dashrightarrow \quad \delta\Omega(t) = \delta\Omega_0 \times e^{-\frac{t}{\tau_{\rm r}}}$$

$\delta\Omega_0 \leftrightarrow$ *trigger threshold*

- Theoretical rise time:

$$\boxed{\tau_{\rm r} = \frac{\bar{I}_{\rm p}}{\bar{I}} \times \frac{1 - \bar{\varepsilon}_p - \bar{\varepsilon}_{\rm n}}{2\zeta\bar{\mathcal{B}}\Omega_{\rm n}}}$$

- Numerical modelling:

Computation of $\Omega_{\rm n}(t)$ & $\Omega_{\rm p}(t)$ profiles from $\Omega_{\rm n,0} > \Omega_{\rm p,0}$

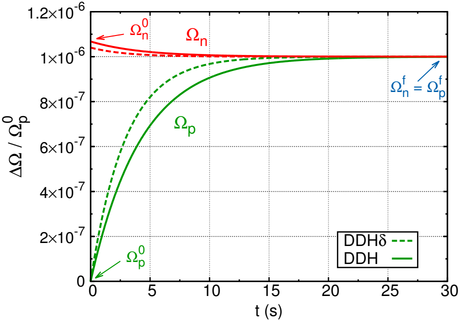
Input parameters

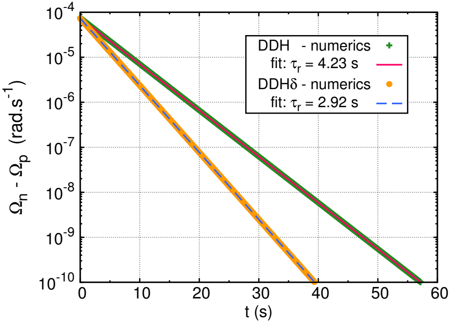
$M^{\rm B}$, Ω, *EoS*, β-eq., $\Delta\Omega/\Omega$, $\overline{\mathcal{B}}$

Time evolution

$$\Delta\Omega/\Omega = 10^{-6},\ \Omega_{\mathrm{n}}^{f}/2\pi = \Omega_{\mathrm{p}}^{f}/2\pi = 11.19\ \mathrm{Hz},$$

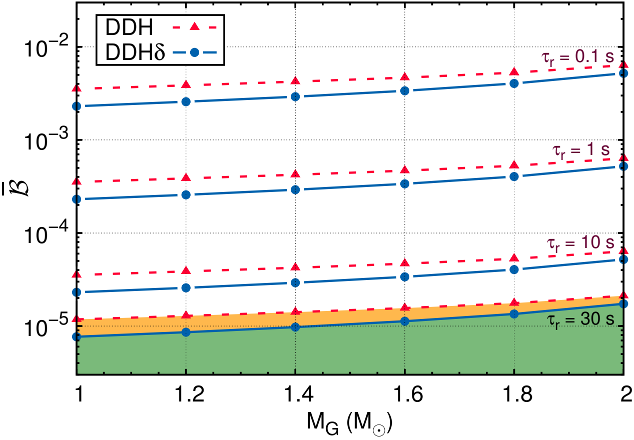
$$M_G = 1.4\ \mathrm{M}_{\odot}\ \&\ \overline{\mathcal{B}} = 10^{-4}$$





The Vela pulsar

$$\Delta\Omega/\Omega = 10^{-6},\ \Omega_{\rm n}^f/2\pi = \Omega_{\rm p}^f/2\pi = 11.19\ \text{Hz}$$



- $\bar{\mathcal{B}} \nearrow \Longrightarrow \tau_{\rm r} \searrow$
- Lower limit:

  $\tau_{\rm r} < 30$ s

  $\Downarrow$

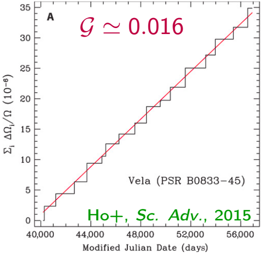
  $\bar{\mathcal{B}} > 10^{-5}$

Additional physical inputs

So far, we assumed that **all** the neutrons can decouple from the protons.

- only a *small fraction* of the neutron fluid could be involved in the glitch:

$$\bar{I}_{\rm n}/\bar{I} > f \equiv I_{\rm n}^{\rm nc}/\bar{I} \gtrsim \mathcal{G} \times (1-\varepsilon_{\rm n}^{\rm nc})$$



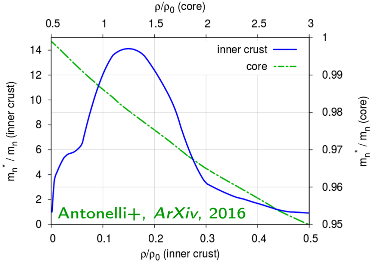
$\mathcal{G} \simeq 0.016$

Ho+, *Sc. Adv.*, 2015

See also: Link+, *PRL*, 1999 & Lyne+, *MNRAS*, 2000

- we need to account for *crustal entrainment* (Bragg scattering):

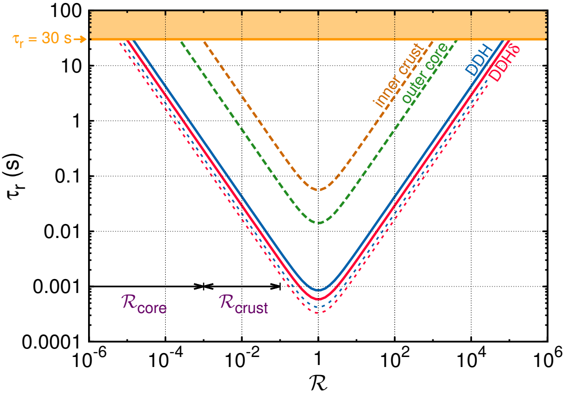
$$-14 \lesssim \varepsilon_{\rm n}^{\rm nc} \lesssim 0$$



Antonelli+, *ArXiv*, 2016

See also: Chamel, *PRC*, 2012

Constraining the interior of NSs



$$\boxed{\tau_{\rm r} = \frac{1 - f - \varepsilon_{\rm n}^{\rm nc}}{2\bar{\mathcal{B}}\Omega_{\rm n}}}$$

- whole core:

  $f = 0.94, \ \varepsilon_{\rm n}^{\rm nc} = 0.03$ (DDH)
  $f = 0.96, \ \varepsilon_{\rm n}^{\rm nc} = 0.02$ (DDHδ)

- outer core:

  $f = 0.016, \ \varepsilon_{\rm n}^{\rm nc} = 0$

- inner crust:

  $f = 0.064, \ \varepsilon_{\rm n}^{\rm nc} = -3$

$\tau_{\rm r}^{min} \simeq$ 1 ms - 0.1 s $\dashrightarrow$ the glitch event is a quasi-stationary process ✓

Conclusion & perspectives

- *Additional coupling* through the relativistic frame-dragging effect,
- *Relativistic corrections* on the spin-up time: $\sim 50\%$ (core),

$\hookrightarrow$ should be included in a quantitative model of glitches.

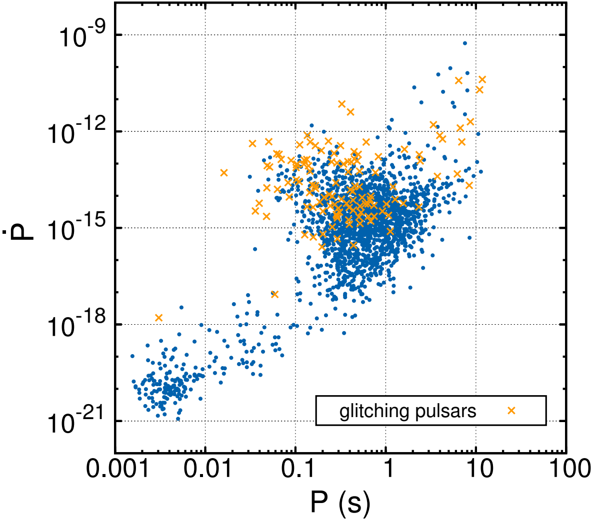
Future work:

- Improve our numerical models by including the crust and considering that only a small amount of vortices is involved in the glitch event,
- Comparison with future accurate observations of glitches,
- Build a complete model of glitch (incl. relaxation) $\leftrightarrow \bar{\mathcal{B}}(t)$.

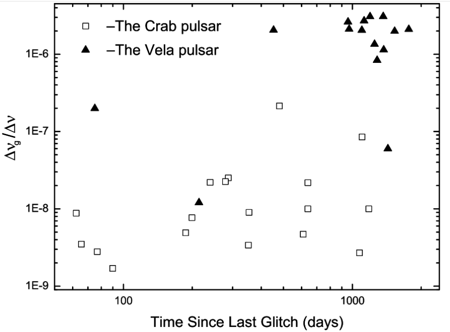
Thank you!

$P - \dot{P}$ diagram

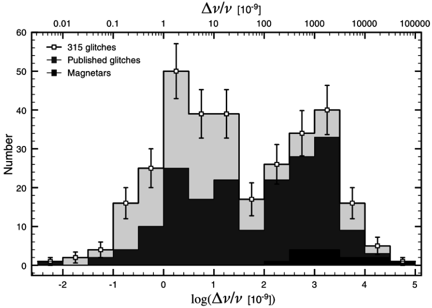
ATNF Pulsar Database ; Manchester et al., *Astron. Journal*, 2005



Different kinds of glitches ?



Wang et al., *Ap& SS*, 2012



Espinoza et al., *MNRAS*, 2011

Glitch activity

Observables

Link, Epstein & Lattimer, *PRL*, 1999

- Average glitch activity:

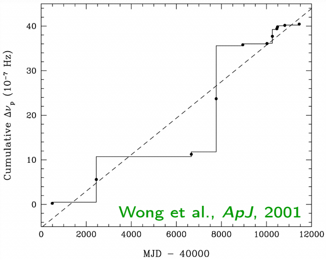
$$\overline{\mathrm{A}} = \frac{1}{t_{\mathrm{obs}}} \frac{\sum_i \Delta\Omega_i}{\Omega}$$

- Coupling parameter:

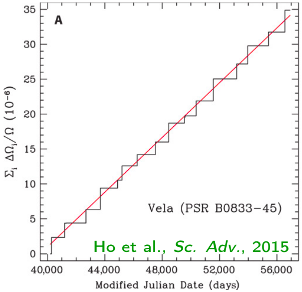
$$\boxed{\mathrm{G} = \frac{\Omega}{|\dot{\Omega}|} \times \overline{\mathrm{A}}}$$

⇢ **Vela**: $\mathrm{G} \simeq 1.62 \times 10^{-2}$

⇢ **Crab**: $\mathrm{G} \simeq 1.45 \times 10^{-5}$



Wong et al., *ApJ*, 2001



Ho et al., *Sc. Adv.*, 2015

Spacetime metric

Bonazzola, Gourgoulhon, Salgado & Marck, *A&A*, 1993

Rotating neutron stars, at **equilibrium**, described by $(\mathcal{E}, \boldsymbol{g})$:

- asymptotically flat: $\boldsymbol{g} \to \boldsymbol{\eta}$ at spatial infinity $(r \to +\infty)$,
- stationary & axisymmetric: $\frac{\partial g_{\alpha\beta}}{\partial t} = \frac{\partial g_{\alpha\beta}}{\partial \varphi} = 0$,
- circular: perfect fluids $\Rightarrow$ *purely circular* motion around the rotation axis with Ω_{n}, Ω_{p} (+ **rigid rotation**).

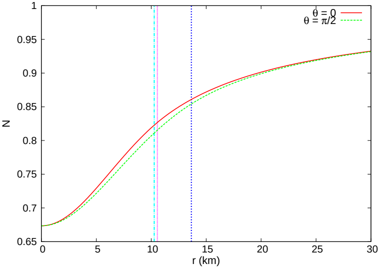
Spacetime metric in quasi-isotropic coordinates:

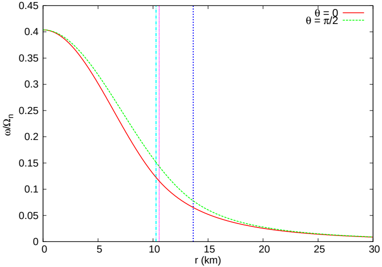
$$g_{\alpha\beta}\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta = -N^2\,\mathrm{d}t^2 + A^2(\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2) + B^2 r^2 \sin^2\theta(\mathrm{d}\varphi - \omega\,\mathrm{d}t)^2$$

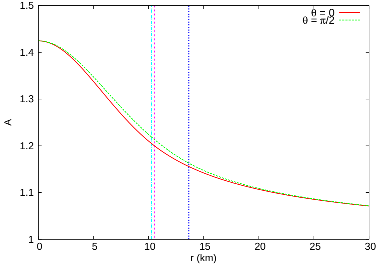
At spatial infinity

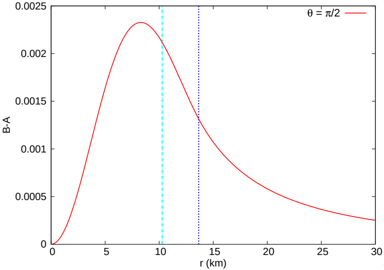
$$N, A, B \to 1 \quad \& \quad \omega \to 0$$

Metric potentials









Relativistic two-fluid hydrodynamics

Carter, "Covariant theory of conductivity in ideal fluid or solid media", 1989 & Carter & Langlois, *Nuc. Phys. B*, 1998

System = two **perfect** fluids:

- superfluid neutrons → $\vec{n}_{\rm n} = n_{\rm n}\vec{u}_{\rm n}$,
- protons & electrons → $\vec{n}_{\rm p} = n_{\rm p}\vec{u}_{\rm p}$.

Energy-momentum tensor

$$T_{\alpha\beta} = n_{\rm n\alpha} p^{\rm n}_{\beta} + n_{\rm p\alpha} p^{\rm p}_{\beta} + \Psi g_{\alpha\beta}$$

↪ **conjugate momenta**

Entrainment matrix:

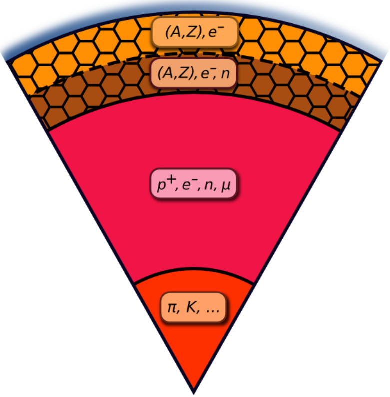
$$\begin{cases} p^{\rm n}_{\alpha} & = \mathcal{K}^{\rm nn} n^{\rm n}_{\alpha} + \mathcal{K}^{\rm np} n^{\rm p}_{\alpha} \\ p^{\rm p}_{\alpha} & = \mathcal{K}^{\rm pn} n^{\rm n}_{\alpha} + \mathcal{K}^{\rm pp} n^{\rm p}_{\alpha} \end{cases}$$

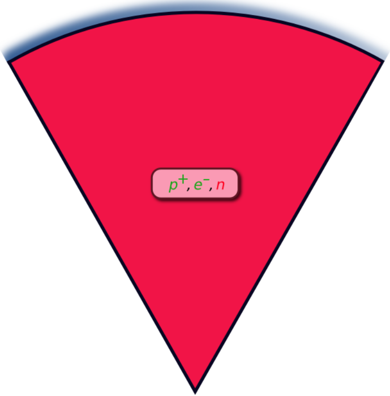
⇢ entrainment effect

Equation of state

$$\mathcal{E}(n_{\rm n}, n_{\rm p}, \Delta^2)$$

Neutron stars interior

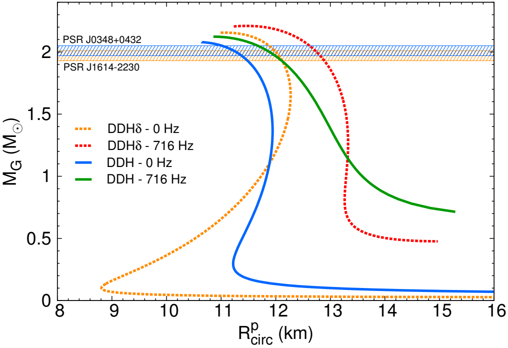




Equations of state

Relativistic Mean-Field Theory:

strong interaction between nucleons ⇔ exchange of effective mesons



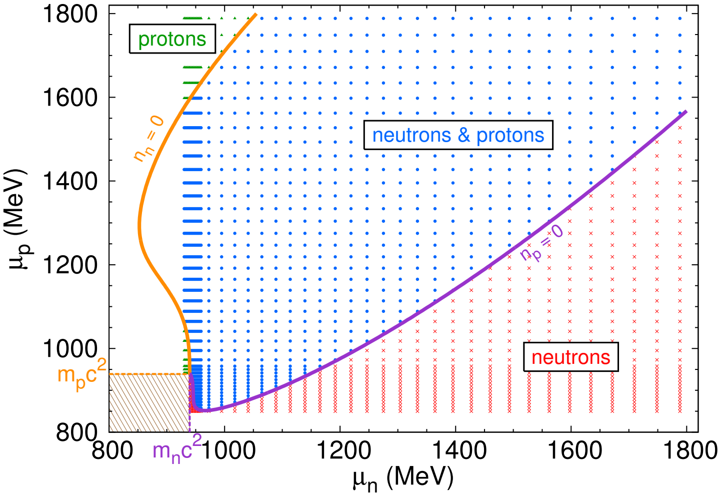
- Gravitational mass:

$$M_G = M^B + E_{\rm bind},$$

- Circumferential radius:

$$R^X_{\rm circ,\ eq} = \mathcal{C}^X / 2\pi.$$

Tabulated EoS



Entrainment effects

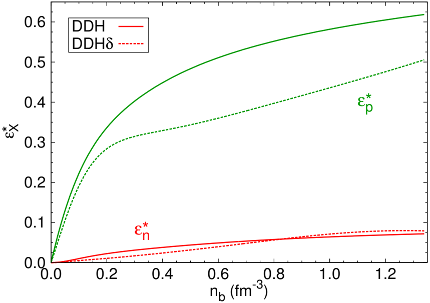
Dynamical effective mass:

$$\boxed{{}^3\vec{\boldsymbol{p}}_{\boldsymbol{X}} = m_X^* \; {}^3\vec{\boldsymbol{u}}_{\boldsymbol{X}}}$$

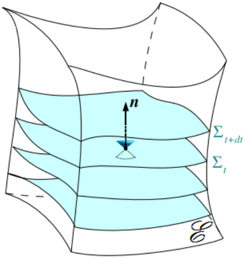
→ in the *rest frame* of the second fluid.

Zero-velocity frame:

$$m_X^* = \overbrace{\mu^X}^{\text{special relativity}} \times \left(1 - \underbrace{\varepsilon_X^*}_{\textbf{entrainment}}\right)$$



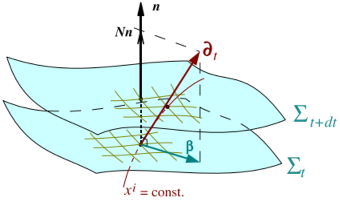
3+1 formalism



Foliation of the spacetime $(\mathcal{E}, \boldsymbol{g})$ by $(\Sigma_t)_{t\in\mathbb{R}}$, with unit normal $\vec{\boldsymbol{n}}$

Eulerian observer $\mathcal{O}_n$: 4-velocity $= \vec{\boldsymbol{n}}$

- lapse function N : $\vec{\boldsymbol{n}} = -N\vec{\boldsymbol{\nabla}}t$,
- shift vector $\vec{\beta}$: $\vec{\boldsymbol{\partial}}_t = N\vec{\boldsymbol{n}} + \vec{\beta}$.



3+1 metric:

$$g_{\alpha\beta}\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta = -N^2\,\mathrm{d}t^2 + \gamma_{ij}\left(\mathrm{d}x^i + \beta^i\,\mathrm{d}t\right)\left(\mathrm{d}x^j + \beta^j\,\mathrm{d}t\right)$$

Numerical procedure

Paramètres d'entrée :

- une EOS
- H_c^n, H_c^p
- Ω_n, Ω_p

$\boldsymbol{i = 0}$

Initialisation :

- $N = A = B = 1$ et $\omega = 0, \forall\, (r, \theta)$
- $U_n = U_p = 0$
- $H_0^i(r,\theta) = H_c^i \left(1 - \frac{r^2}{R^2}\right)$

Convergence threshold

$$|H_{k+1}^i(r,\theta) - H_k^i(r,\theta)| < \epsilon$$

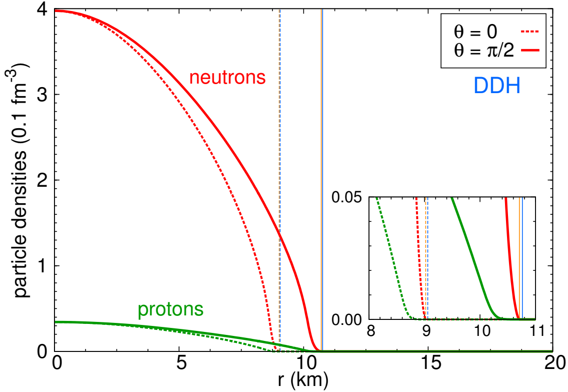
At each iteration

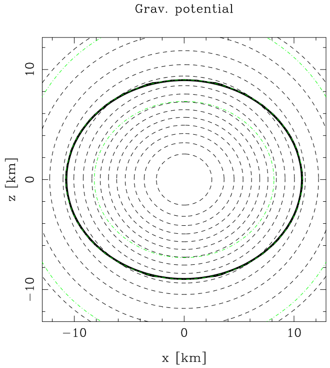
For given values of $(\mu^{\mathrm{n}}, \mu^{\mathrm{p}}, \Delta^2)$, we compute:

1. Ψ, n_{n}, n_{p} and α from the EoS
2. The source terms E, p_φ, ${S^i}_i$,
3. Einstein Equations are solved,
4. Kinetic terms U_{i} et Γ_{i},
5. Computation of H_{k+1}^i.

Density profiles

$$M_G = 1.4\ \mathrm{M}_\odot,\ \Omega_\mathrm{n}/2\pi = \Omega_\mathrm{p}/2\pi = 716\ \mathrm{Hz}$$



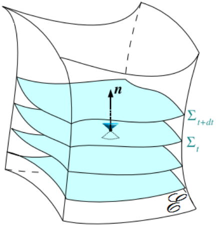


Angular momenta

Axisymmetry $\leftrightarrow$ $\vec{\chi}$

Komar definition:

$$J_{\mathrm{K}} = -\int_{\Sigma_t} \underbrace{\boldsymbol{T}(\vec{\boldsymbol{n}},\vec{\chi})}_{-p_\varphi} \; \mathrm{d}^3 V$$



Eulerian observer $\vec{\boldsymbol{n}}$ (3+1)

Angular momentum of each fluid

Langlois, Sedrakian & Carter, *MNRAS*, 1998

$$p_\varphi = \underbrace{\Gamma_{\mathrm{n}} n_{\mathrm{n}} p^{\mathrm{n}}_\varphi}_{j^{\mathrm{n}}_\varphi} + \underbrace{\Gamma_{\mathrm{p}} n_{\mathrm{p}} p^{\mathrm{p}}_\varphi}_{j^{\mathrm{p}}_\varphi}$$

$$J_X = \int_{\Sigma_t} j^X_\varphi \; A^2 B r^2 \sin\theta \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}\varphi$$

Fluid couplings

For $\Omega_{\rm n} \simeq \Omega_{\rm p}$, the **angular momentum of fluid** X reads

$$\begin{aligned} J_X &\simeq \int_{\Sigma_t} \Gamma_X^2 n_X \mu^X \frac{B}{N} (\Omega_X - \omega)\, r^2 \sin^2\theta \ \mathrm{d}^3V \\ &+ \int_{\Sigma_t} \Gamma_X^2 n_X \mu^X \varepsilon_X \frac{B}{N} (\Omega_Y - \Omega_X)\, r^2 \sin^2\theta \ \mathrm{d}^3V \end{aligned}$$

Introducing $i_X \equiv \Gamma_X^2 n_X \mu^X \frac{B}{N} r^2 \sin^2\theta$, we characterize the couplings by

- Entrainment:
- Lense-Thirring:

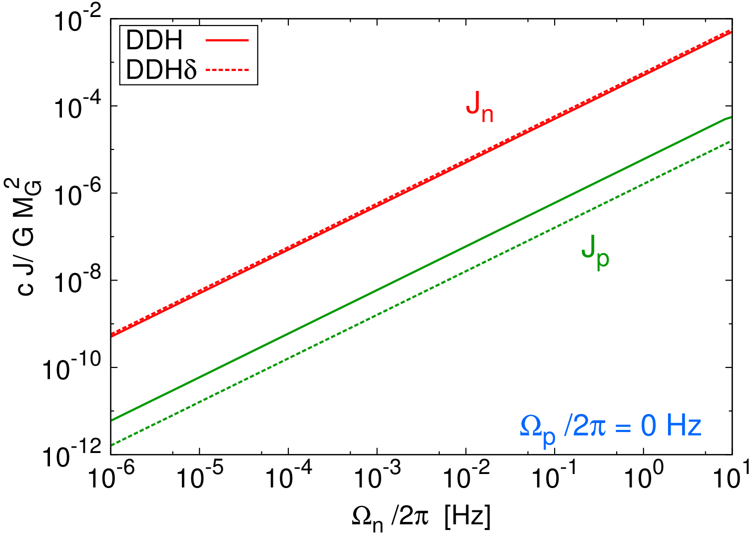
$$\boxed{\tilde{I}_X \ \varepsilon_X^{entr} \equiv \int_{\Sigma_t} i_X \ \varepsilon_X \ \mathrm{d}^3V}$$

$$\boxed{\tilde{I}_X \left(\varepsilon^{LT}_{X\to X}\Omega_X + \varepsilon^{LT}_{Y\to X}\Omega_Y\right) \equiv \int_{\Sigma_t} i_X \ \omega \ \mathrm{d}^3V}$$

where $\tilde{I}_X \equiv \int_{\Sigma_t} i_X \ \mathrm{d}^3V$

$$J_X = \tilde{I}_X \left(1 - \varepsilon^{LT}_{X\to X} - \varepsilon_X^{entr}\right)\Omega_X + \tilde{I}_X \left(\varepsilon_X^{entr} - \varepsilon^{LT}_{Y\to X}\right)\Omega_Y$$

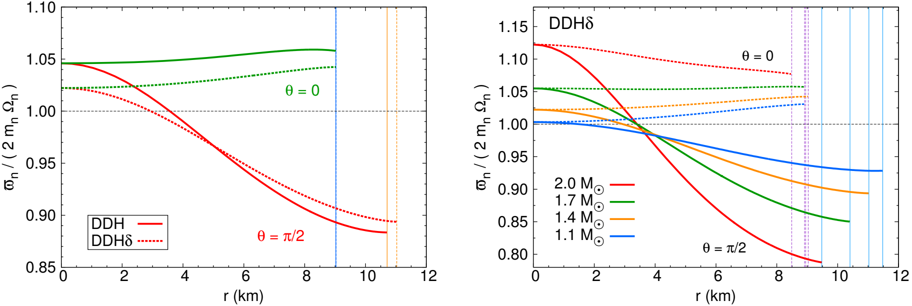
Fluid couplings



Vorticity

Superfluid vorticity

$$w_{\mu\nu} = \nabla_\mu p^{\rm n}_\nu - \nabla_\nu p^{\rm n}_\mu \quad \dashrightarrow \quad \varpi_{\rm n} = \sqrt{\frac{w_{\mu\nu} w^{\mu\nu}}{2}}$$



$$\Omega^{\rm n}/2\pi = \Omega^{\rm p}/2\pi = 716 \text{ Hz}$$

Chemical equilibrium

Typical timescales

Yakovlev et al., *Physical Reports*, 2001

- **Direct Urca** :

$$\tau_\beta \simeq 20 \left(\frac{T}{10^9 \text{ K}}\right)^{-4} \text{ s}$$

- **Modified Urca** :

$$\tau_\beta \simeq \left(\frac{T}{10^9 \text{ K}}\right)^{-6} \text{ months}$$

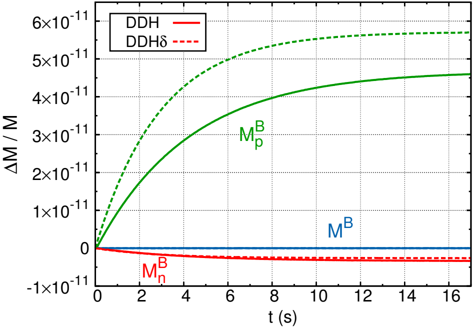
Two limiting cases:

- $\tau_\beta = 0 \dashrightarrow \beta$-eq. is enforced.

$$\boxed{M^{\text{B}} = \text{cte} \;\; \& \;\; \mu_c^{\text{n}} = \mu_c^{\text{p}}}$$

- $\tau_\beta \to \infty \dashrightarrow$ no reactions.

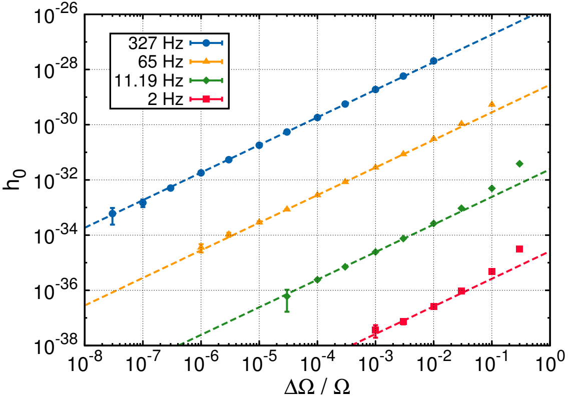
$$\boxed{M_{\text{n}}^{\text{B}} = \text{cte} \;\; \& \;\; M_{\text{p}}^{\text{B}} = \text{cte}}$$



$$\Delta\Omega/\Omega = 10^{-4},\ \Omega_f/2\pi = 327 \text{ Hz, DDH,}$$
$$M_G = 1.4 \text{ M}_\odot,\ \mathcal{B} = 10^{-4}$$

Gravitational wave amplitude

$$h_+(t) = -\frac{3}{2}\sin^2 i\,\frac{\mathcal{G}}{Dc^4}\ddot{Q} = h_0 \sin^2 i\;e^{-\frac{t}{\tau_r}}$$

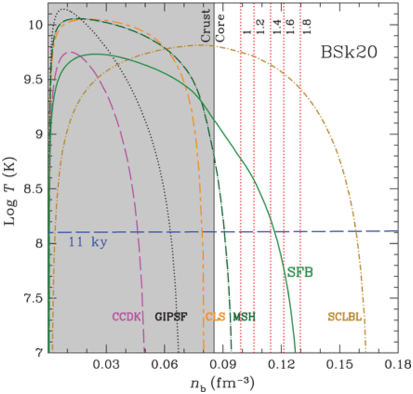


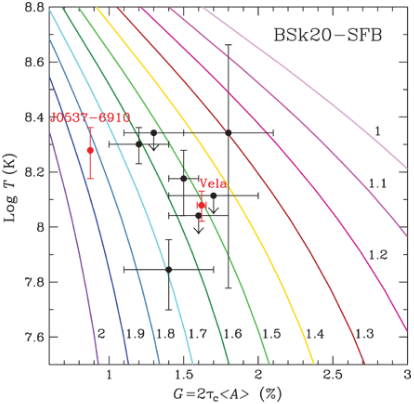
- $D = 1$ kpc,
- $\overline{\mathcal{B}} = 10^{-3}$,
- $M_G = 1.4\ \mathrm{M}_\odot$,
- DDH EoS.

$$h_0 \simeq 1.0 \times 10^{-37}\left(\frac{D}{1\ \mathrm{kpc}}\right)^{-1}\left(\frac{\bar{\mathcal{B}}}{10^{-3}}\right)^{2}\left(\frac{\Omega}{10^{2}\ \mathrm{rad.s}^{-1}}\right)^{4}\left(\frac{\Delta\Omega/\Omega}{10^{-6}}\right)$$

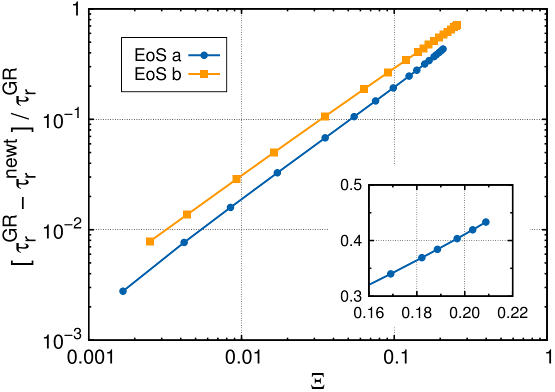
Measuring mass using pulsar glitches

Ho, Espinoza, Antonopoulou & Andersson, *Sci. Adv.*, 2015





Influence of general relativity



- polytropic EoSs
- **compactness** parameter:

$$\boxed{\Xi = \frac{G\,M_G}{R_{c,eq}\,c^2}}$$

NB: for NSs, $\Xi \simeq 0.2$

$\longrightarrow$ rise time longer within the general relativistic framework.