Electrical conductivity of warm neutron star crusts in magnetic fields

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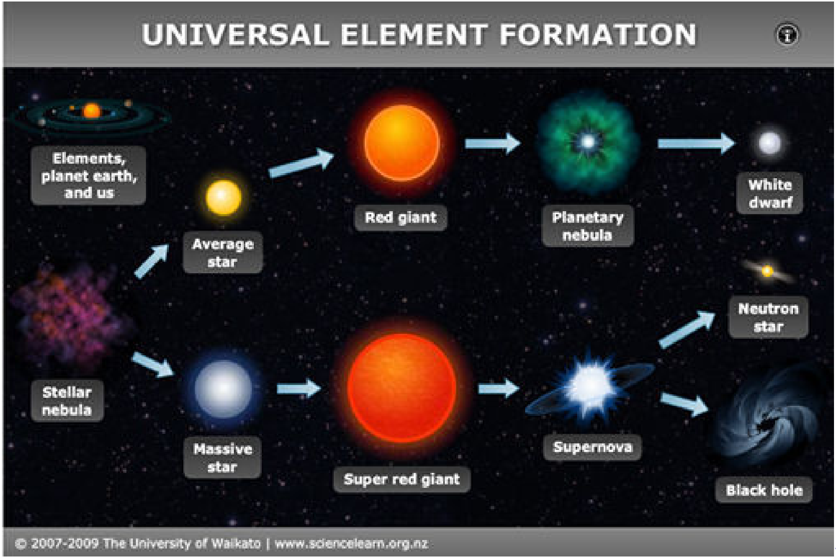
Structure of this talk

- Introduction and motivation
- Kinetic theory
- Relaxation time
- Results
- Summary

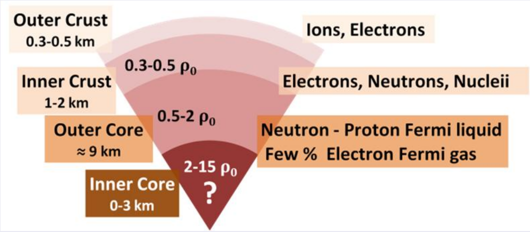
Neutron stars



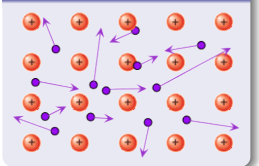
The inner structure of a neutron star

Neutron stars (NS) are compact stellar objects with masses $M \sim M_\odot$, radii $R \sim 10$ km and magnetic fields $B \sim 10^{10} - 10^{14}$ G.

The inner structure of a NS



The composition of the outer crust



The outer crust of the star consists of nuclei and conducting Fermi gas of relativistic electrons.

Transport coefficients

Transport coefficients play important role in several astrophysical problems:

- Cooling of a NS
- Magnetic field evolution
- Electromagnetic radiation from the surface of a NS
- Radial oscillations of NS surface
- Propagation of magneto-hydrodynamic (MHD) waves in NS crust

Transport properties of NS crust are well studied at low temperatures (T <1 MeV). More recently, resistive MHD simulations of relativistic stellar systems, in particular of binary magnetized NS mergers and the hypermassive NS has brought into focus the treatment of warm (heated) crustal matter. Such matter is also expected in proto-neutron stars newly formed in the aftermath of supernova explosion and in accreting neutron stars. The aim of this work is the calculation of the electrical conductivity of warm outer crust of a NS. We calculate the electrical conductivity for densities $10^6 < \rho < 10^{11}$ g cm^{-3}, temperatures $0.1 < T < 10$ MeV and non-quantizing magnetic fields $10^{10} < B < 10^{14}$ G. The density-temperature range discussed here covers the degenerate regime as well as the non-degenerate regime for electrons.

The Boltzmann equation

The kinetics of electrons is described by the Boltzmann equation for the distribution function

$$\frac{\partial f}{\partial t} + \boldsymbol{v}\frac{\partial f}{\partial \boldsymbol{r}} - e(\boldsymbol{E} + [\boldsymbol{v} \times \boldsymbol{H}])\frac{\partial f}{\partial \boldsymbol{p}} = I[f].$$

The collision integral for electron-ion scattering has the form

$$I = -(2\pi)^4 \sum_{234} |\mathcal{M}_{12\to 34}|^2 \delta^{(4)}(p_1 + p_2 - p_3 - p_4)[f_1(1-f_3)g_2 - f_3(1-f_1)g_4].$$

For small perturbations the solution of the Boltzmann equation can be searched in the form

$$f = f_0 + \delta f, \quad f_0 = \frac{1}{e^{\beta(\varepsilon-\mu)} + 1}, \quad g = n_i(2\pi\beta/M)^{3/2} e^{-\beta\varepsilon}.$$

In the relaxation time approximation the solution has the form

$$\delta f = \frac{e\tau}{1 + (\omega_c\tau)^2}\frac{\partial f_0}{\partial \varepsilon} v_i\left[\delta_{ij} - \omega_c\tau\varepsilon_{ijk}h_k + (\omega_c\tau)^2 h_i h_j\right]E_j, \quad h_i = H_i/H.$$

The relaxation time is given by the formula

$$\tau^{-1} = (2\pi)^{-5}\int d\omega d\boldsymbol{q} d\boldsymbol{p}_2 \frac{\boldsymbol{q}\cdot\boldsymbol{p}}{p^2}|\mathcal{M}_{12\to 34}|^2 \delta(\varepsilon_1 - \varepsilon_3 - \omega)\delta(\varepsilon_2 - \varepsilon_4 + \omega) g_2 (1 - f_3^0)(1 - f_1^0)^{-1}.$$

The electrical conductivity tensor

The electrical conductivity tensor is defined as the coefficient of proportionality between the electrical current and electrical field

$$j_i = -\int \frac{2d\boldsymbol{p}}{(2\pi)^3} e v_i \delta f = \sigma_{ij} E_j.$$

If magnetic field is directed along z axis σ_{ij} tensor has the form

$$\hat{\sigma} = \begin{pmatrix} \sigma_0 & -\sigma_1 & 0 \\ \sigma_1 & \sigma_0 & 0 \\ 0 & 0 & \sigma \end{pmatrix}.$$

- Longitudinal conductivity does not depend on magnetic field (isotropic conduction)

$$\sigma = -\frac{e^2}{3\pi^2} \int_0^\infty dp p^2 v^2 \tau \frac{\partial f_0}{\partial \varepsilon}.$$

- Transversal (σ_0) and Hall (σ_1) conductivities are magnetic field dependent

$$\sigma_0 = -\frac{e^2}{3\pi^2} \int_0^\infty dp p^2 v^2 \frac{\tau}{1+(\omega_c\tau)^2} \frac{\partial f_0}{\partial \varepsilon}, \qquad \sigma_1 = -\frac{e^2}{3\pi^2} \int_0^\infty dp p^2 v^2 \frac{\omega_c\tau^2}{1+(\omega_c\tau)^2} \frac{\partial f_0}{\partial \varepsilon}.$$

The two components σ_0 and σ_1 depend on magnetic field via the dimensionless product $\omega_c\tau$, where $\omega_c = eB/\varepsilon$ is the cyclotron frequency of electrons.

Recovering limiting cases

For two limiting cases of strongly degenerate and non-degenerate electrons the following analytical formulae can be obtained:

- At low temperature limit $T \ll T_F$ doing the substitution $\partial f^0/\partial\varepsilon \to -\delta(\varepsilon - \varepsilon_F)$ one comes to the famous Drude formulae

$$\sigma = \frac{n_e e^2 \tau_F}{\varepsilon_F}, \quad \sigma_0 = \frac{\sigma}{1+(\omega_{cF}\tau_F)^2}, \quad \sigma_1 = \frac{\omega_{cF}\tau_F}{1+(\omega_{cF}\tau_F)^2}\sigma.$$

- At high temperature limit $T \gg T_F$ Drude-type formulae work with 20% precision ($\bar{\varepsilon} \simeq 3T$ is the average energy of ultrarelativistic electrons in the non-degenerate regime)

$$\sigma \simeq \frac{n_e e^2 \bar{\tau}}{\bar{\varepsilon}}, \quad \sigma_0 \simeq \frac{\sigma}{1+(\bar{\omega}_c\bar{\tau})^2}, \quad \sigma_1 \simeq \frac{\bar{\omega}_c\bar{\tau}}{1+(\bar{\omega}_c\bar{\tau})^2}\sigma.$$

The anisotropy of the conductivity tensor depends on the value of the parameter $\omega_c\tau$:

- If $\omega_c\tau \ll 1$ (weak magnetic fields), $\sigma_0 \simeq \sigma$, $\sigma_1 \simeq \omega_c\tau\sigma \ll \sigma$ and the conduction is isotropic

$$\sigma_{kj} \simeq \delta_{kj}\sigma.$$

- If $\omega_c\tau \gg 1$ (strong magnetic fields), $\sigma_0 \simeq \sigma(\omega_c\tau)^{-2} \ll \sigma$, $\sigma_1 \simeq \sigma(\omega_c\tau)^{-1} \ll \sigma$ and the conductivities transversal to magnetic field are strongly suppressed.

The electron scattering mechanism

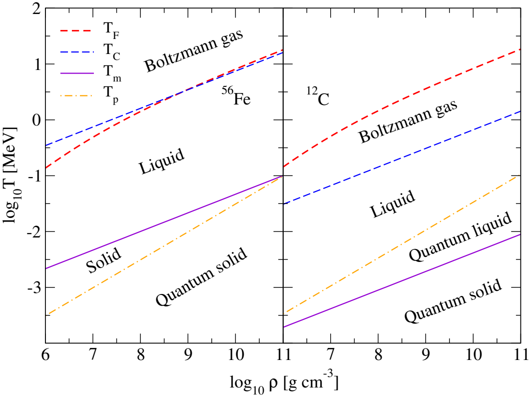
The electron scattering mechanism depends on the state of nuclei. The latter is controlled by the value of the plasma parameter Γ.

- If $\Gamma < 1$ - Boltzmann gas, the scattering is on individual, uncorrelated nuclei
- If $1 < \Gamma < \Gamma_m \simeq 160$ - liquid state, the scattering is on correlated nuclei
- If $\Gamma > \Gamma_m$ - crystal state, the scattering is on phonons and impurities

$$\Gamma = \frac{Z^2 e^2}{a_i T}, \quad a_i = \left(\frac{3}{4\pi n_i}\right)^{1/3}$$

$$T_C = \frac{Z^2 e^2}{a_i}, \quad T_m = \frac{Z^2 e^2}{a_i \Gamma_m}$$

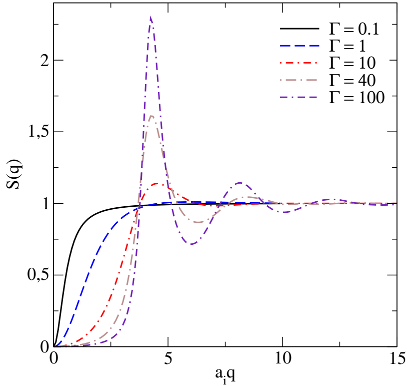
$$T_p = \left(\frac{4\pi Z^2 e^2 n_i}{M}\right)^{1/2}$$



In the case of liquid plasma the conductivity is dominated by the electron scattering off correlated nuclei. The correlation is taken into account via 2-point structure factor for nuclei.

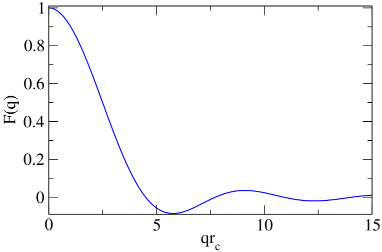
The ion-ion correlation function and nuclear formfactor

The ion-ion interaction is taken into account via 2-point structure function $S(q)$, which depends on the value of the parameter Γ. We assume that only one sort of ions exists at a given density, so that the structure functions of one-component plasma (OCP) can be used. We adopt the Monte-Carlo results of Galam and Hansen ($\Gamma \geq 2$) and the analytical (leading order) expressions derived by Tamashiro et al. ($\Gamma \leq 2$).



$$F(q) = -3\frac{qr_c\cos(qr_c) - \sin(qr_c)}{(qr_c)^3}$$

$$r_c = 1.15\,A^{1/3}\mathrm{fm}$$

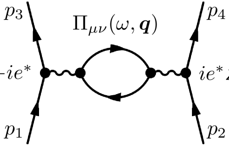


The ion structure factor suppresses the scattering with small q. In $a_i q \gg 1$ limit $S(q) \to 1$.

The scattering probability

The electron-ion scattering matrix element can be calculated according to the Feynman rules

$$\mathcal{M}_{12\to 34} = -ie^*$$



$$ie^* Z = J_\mu D^{\mu\nu} J'_\nu = \frac{J_0 J'_0}{q^2 + \Pi_l} - \frac{\boldsymbol{J}_t \boldsymbol{J}'_t}{q^2 - \omega^2 + \Pi_t}$$

J^μ, J'^μ are electronic and ionic 4-currents respectively

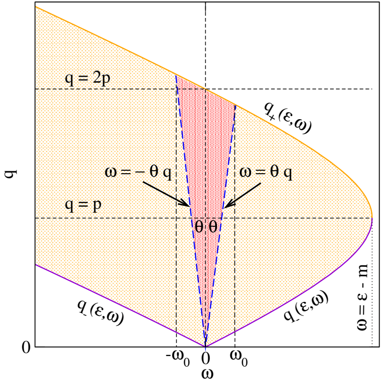
$$J^\mu = -e^* \bar{u}^{s_3}(p_3)\gamma^\mu u^{s_1}(p_1), \qquad J'^\mu = Ze^* v_2^\mu .$$

The relaxation time in the case of inelastic electron-ion scattering is given by the formula

$$\tau^{-1}(\varepsilon) = \frac{\pi Z^2 e^4 n_i}{\varepsilon p^3} \int_{-\infty}^{\varepsilon - m} d\omega e^{-\omega/2T} \frac{f^0(\varepsilon - \omega)}{f^0(\varepsilon)} \int_{q_-}^{q_+} dq (q^2 - \omega^2 + 2\varepsilon\omega) S(q) F^2(q) \times$$
$$\times \frac{1}{\sqrt{2\pi}\theta} e^{-\omega^2/2q^2\theta^2} e^{-q^2/8MT} \left\{ \frac{(2\varepsilon - \omega)^2 - q^2}{|q^2 + \Pi_l|^2} + \theta^2 \frac{(q^2 - \omega^2)[(2\varepsilon - \omega)^2 + q^2] - 4m^2 q^2}{q^2 |q^2 - \omega^2 + \Pi_t|^2} \right\}$$

$$e^* = \sqrt{4\pi} e, \quad \theta = \sqrt{T/M}, \quad q_\pm = \left| \pm\sqrt{\varepsilon^2 - m^2} + \sqrt{(\varepsilon - \omega)^2 - m^2} \right|$$

Limit of elastic scattering



Due to the suppression factor θ the scattering is more effective via exchange of virtual photons with energies and momenta lying inside the triangle $\omega/q < \theta$ on the plane (ω, q) (slightly inelastic scattering). In the limit of infinite massive ions ($\theta \to 0$) the electron-ion interaction is pure electrostatic, therefore the scattering becomes elastic

$$\frac{1}{\theta\sqrt{2\pi}} e^{-x^2/2\theta^2} \to \delta(x), \quad \text{as} \quad \theta \to 0$$

In the limit of elastic scattering one comes to the famous formula for the relaxation time

$$\tau_F^{-1} = \frac{4Ze^4\varepsilon_F}{3\pi} \int_0^{2p_F} dq\, q^3 \frac{S(q)F^2(q)}{|q^2 + \Pi_l|^2} \left(1 - \frac{q^2}{4\varepsilon_F^2}\right)$$

The photon polarization tensor

The full photon propagator $D_{\mu\nu}$ can be found from the Dyson equation

$$[D^{-1}]_{\mu\nu} = [D_0^{-1}]_{\mu\nu} - \Pi_{\mu\nu}, \qquad D_0^{\mu\nu} = g^{\mu\nu}/(\omega^2 - q^2).$$

The photon polarization tensor $\Pi_{\mu\nu}$ is decomposed into transvers and longitudinal modes

$$\Pi_{\mu\nu} = \Pi_t P^t_{\mu\nu} + \Pi_l P^l_{\mu\nu}.$$

The formal solution of the Dyson equation can be written as

$$D^{\mu\nu}(q,\omega) = \frac{1}{\omega^2 - q^2}\left[g^{\mu\nu} + \frac{\Pi_t}{\omega^2 - q^2 - \Pi_t} P^{t\mu\nu} + \frac{\Pi_l}{\omega^2 - q^2 - \Pi_l} P^{l\mu\nu}\right].$$

The polarization tensor in the Matsubara space-time at one-loop order is given by the formula

$$\Pi^M_{\mu\nu}(\boldsymbol{q}, i\omega_n) = 4\pi\alpha \int \frac{d\boldsymbol{p}}{(2\pi)^3} T \sum_{m\in Z} \mathrm{Tr}\left[\gamma_\mu S(\boldsymbol{p}, i\omega_m)\gamma_\nu S(\boldsymbol{p}-\boldsymbol{q}, i\omega_m - i\omega_n)\right].$$

The screening of longitudinal and transversal interactions is determined by the corresponding components of the polarization tensor. Within the HTL (hard-thermal-loop) effective field theory of QED ($q \ll p$) they are given by the formulae ($i\omega_n \to \omega + i\delta$)

$$\Pi_l(q,\omega) = -\left(1 - \frac{\omega^2}{q^2}\right)\frac{4\alpha}{\pi}\int_0^\infty p^2 dp \left[\frac{\partial f^+(\varepsilon)}{\partial\varepsilon} + \frac{\partial f^-(\varepsilon)}{\partial\varepsilon}\right]\left[1 - \frac{\omega\varepsilon}{2pq}\log\frac{\omega\varepsilon + pq}{\omega\varepsilon - pq}\right],$$

$$\Pi_t(q,\omega) = -\frac{2\alpha}{\pi}\int_0^\infty p^2 dp \left[\frac{\partial f^+(\varepsilon)}{\partial\varepsilon} + \frac{\partial f^-(\varepsilon)}{\partial\varepsilon}\right]\left[\frac{\omega^2}{q^2} + \left(\frac{p^2}{\varepsilon^2} - \frac{\omega^2}{q^2}\right)\frac{\omega\varepsilon}{2pq}\log\frac{\omega\varepsilon + pq}{\omega\varepsilon - pq}\right].$$

The photon polarization tensor: low-frequency limit

In the degenerate or ultra-relativistic limits the formulae can be simplified

$$\Pi_l = q_D^2 \left(1 - x^2\right) \left[1 - \frac{x}{2\bar{v}} \log \frac{x+\bar{v}}{x-\bar{v}}\right].$$

$$\Pi_t = \frac{1}{2} q_D^2 \left[x^2 + \left(\bar{v}^2 - x^2\right) \frac{x}{2\bar{v}} \log \frac{x+\bar{v}}{x-\bar{v}}\right].$$

$x = \omega/q$, $\bar{v} = v_F$ in the degenerate and $\bar{v} = 1$ in the ultra-relativistic limits, respectively.

The Debye momentum is given by the formula

$$q_D^2 = -\frac{4\alpha}{\pi} \int_0^\infty p^2 dp \left[\frac{\partial f^+(\varepsilon)}{\partial \varepsilon} + \frac{\partial f^-(\varepsilon)}{\partial \varepsilon}\right].$$

Dropping the contribution of antiparticles we find in the limiting cases of highly degenerate and non-degenerate matter

$$q_D^2 \simeq 4\alpha \begin{cases} p_F\, \varepsilon_F/\pi, & T \ll T_F, \\ \pi\, n_e\, /T, & T \gg T_F. \end{cases}$$

We use low-frequency $x = \omega/q \ll 1$ expansions for the polarization tensor to order $O(x^2)$

$$\mathrm{Re}\Pi_l(q,\omega) = (1 - x^2/\bar{v}^2) q_D^2, \quad \mathrm{Im}\Pi_l(q,\omega) = -\frac{\pi}{2}(x/\bar{v}) q_D^2,$$

$$\mathrm{Re}\Pi_t(q,\omega) = x^2 q_D^2, \quad \mathrm{Im}(q,\omega) = \frac{\pi}{4}(x\bar{v}) q_D^2.$$

Relaxation time and the product $\omega_c \tau$ as functions of density

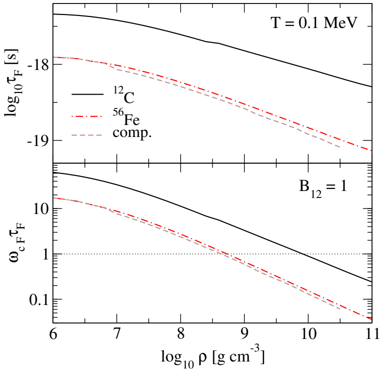
Relaxation time scales as $\tau \propto \varepsilon^{1.8} \rho^{-0.9} T^{-\delta} Z^{-1}$ with $0.1 \leq \delta \leq 0.3$.

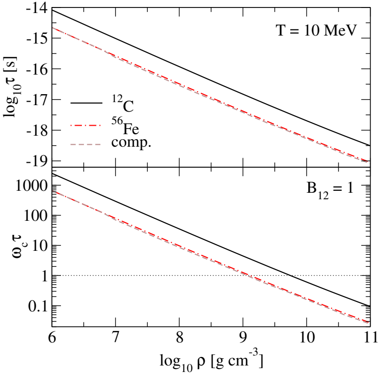
In the degenerate regime

$\varepsilon_F \propto \rho^{1/3}, \quad \tau_F \propto \rho^{-0.3}, \quad \omega_{cF}\tau_F \propto \rho^{-0.6}$

In the non-degenerate regime

$\bar{\varepsilon} \simeq 3T, \quad \bar{\tau} \propto \rho^{-0.9}, \quad \bar{\omega}_c \bar{\tau} \propto \rho^{-0.9}$





Relaxation time decreases faster in the non-degenerate regime.

Relaxation time as a function of temperature in both regimes

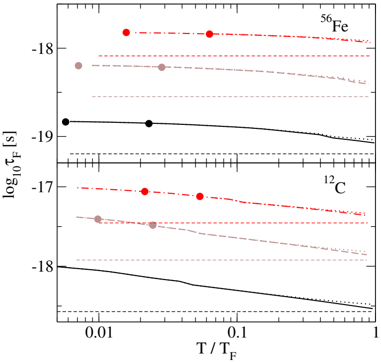
Relaxation time decreases with the temperature in the degenerate and increases in the non-degenerate regime. The decrease in the first case arises solely from the correlation function.

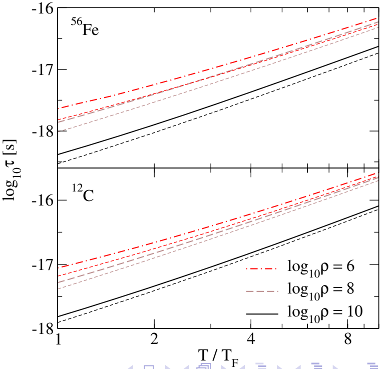
In the degenerate regime

$\varepsilon \to \varepsilon_F, \quad \tau_F \propto T^{-\delta}, \quad \omega_{cF}\tau_F \propto T^{-\delta}$

In the non-degenerate regime

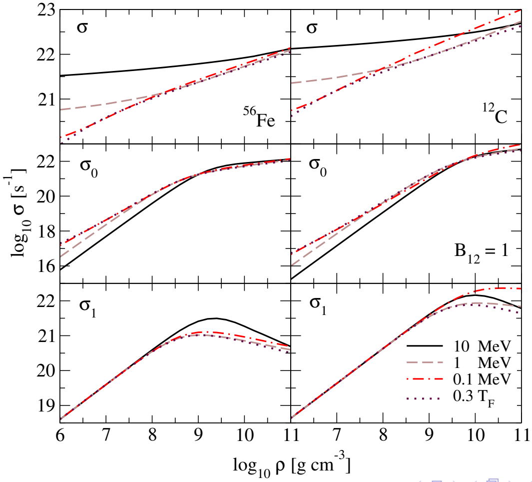
$\bar{\varepsilon} \simeq 3T, \quad \bar{\tau} \propto T^{1.75}, \quad \bar{\omega}_c\bar{\tau} \propto T^{0.75}$





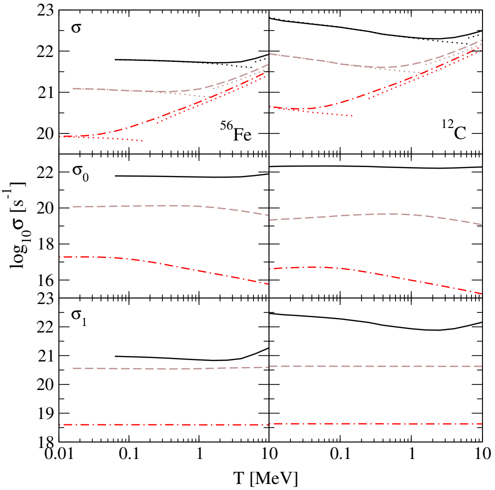
The density dependence of the electrical conductivities

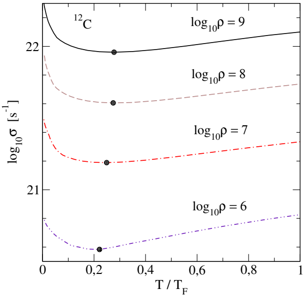
In each regime σ shows a power-law increase with density $\sigma \propto \rho^{\alpha}$.
In the deg. regime $\alpha \simeq 0.4$, in the non-deg. regime $\alpha \simeq 0.1$.



The temperature dependence of the electrical conductivities

σ decreases with temperature in the deg. regime as $\sigma \propto T^{-\delta}$ with $0.1 \leq \delta \leq 0.3$ and increases in the non-deg. regime as $\sigma \propto T^{0.75}$.





The minimum of the conductivity arises at "transition" temperature $T^* \simeq 0.3 T_F$ almost independent on density and type of nuclei.

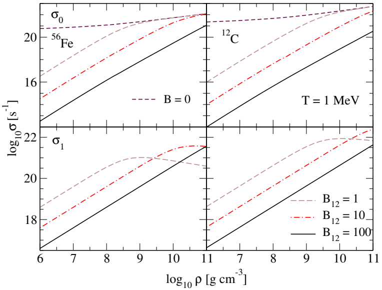
Dependence on magnetic field

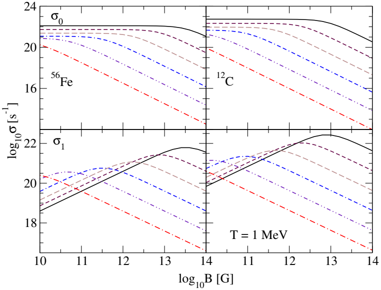
- At high densities or small magnetic fields $\omega_c\tau \ll 1$ (isotropic region) and

$$\sigma_0 \simeq \sigma, \qquad \sigma_1 \simeq \sigma\omega_c\tau \simeq \frac{B}{n_e e}\sigma^2.$$

- At low densities or strong magnetic fields $\omega_c\tau \gg 1$ (strongly anisotropic region) and

$$\sigma_0 \simeq \frac{\sigma}{(\omega_c\tau)^2} \simeq \left(\frac{n_e e}{B}\right)^2 \sigma^{-1}, \qquad \sigma_1 \simeq \frac{\sigma}{\omega_c\tau} \simeq \frac{n_e e}{B}.$$



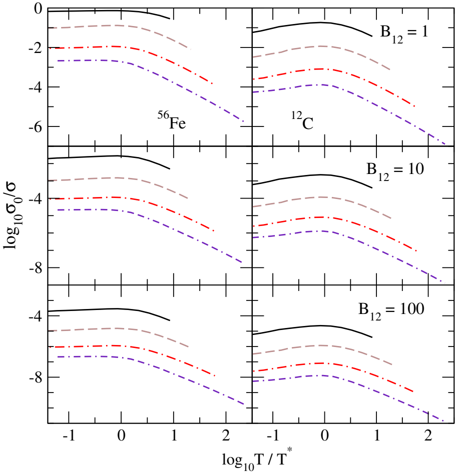


- σ_0 increases with density and decreases with magnetic field.
- The temperature behavior of σ_0 is reversed to that of σ in anisotropic region.
- σ_1 has a maximum at $\omega_c\tau \simeq 1$ as a function of density and magnetic field.
- In anisotropic region σ_1 is independent on temperature and type of nuclei.

The temperature dependence of the crust anisotropy

To characterize the anisotropy we consider the ratio σ_0/σ.

- All curves have a maximum at $T \simeq T^*$ independent of density, magnetic field and type of nuclei.
- At this maximum the anisotropy of the crust is the smallest.
- In the degenerate regime the anisotropy decreases with temperature.
- In the non-degenerate regime $\sigma_0/\sigma \propto T^{-3/2}$ and the crust is strongly anisotropic at very high temperatures.



Density dependent crust composition

The composition of the crust depends on density and temperature. For not very high temperatures the β-equilibrium composition derived for $T = 0$ can be used.

For the conductivity we have the scaling

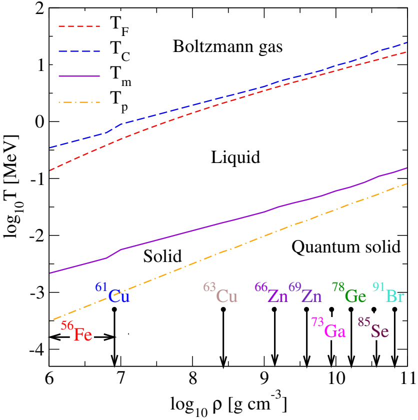
- In the degenerate regime

$$\sigma \propto \frac{n_e \tau_F}{\varepsilon_F} \propto \left(\frac{Z}{A}\right)^{1/3} Z^{-1}.$$

- In the non-degenerate regime

$$\sigma \propto \frac{n_e \bar{\tau}}{\bar{\varepsilon}} \propto Z^{-1}.$$

The results of density dependent composition differ from those of ^{56}Fe less than by a factor 1.4.



It is interesting to study the conductivity of warm matter, which is composed of nuclei in statistical equilibrium, in which case the crust composition may become an important factor.

Fit formulae for three components of the conductivity tensor

We have performed fit to the first component of the conductivity tensor using the formula

$$\sigma^{fit} = CT_F^a \left(\frac{T}{T_F}\right)^{-b} \left(\frac{T}{T_F} + d\right)^{b+c}$$

For the other two components the following fit formulae can be used

$$\sigma_0^{fit} = \frac{\sigma'}{1 + \delta^2 \sigma'^2}, \qquad \sigma_1^{fit} = \frac{\delta \sigma''^2}{1 + \delta^2 \sigma''^2}.$$

$$\delta = B(n_e ec)^{-1}, \quad \sigma' = \sigma^{fit}(T_F/\varepsilon_F)^g, \quad \sigma'' = \sigma^{fit}(1 + T/T_F)^h.$$

- The form of the fit formulae reproduces the correct temperature and density dependence of the conductivity in limiting cases of strongly degenerate and non-degenerate electrons, as well as in limiting cases of isotropic and strongly anisotropic conduction.
- The fit parameters C, a, b, c, d, g, h depend on the ionic structure of the material.
- The relative error of the fit formulae vary from 7% to 15% depending on the composition.

Summary

Summary

- The electrical conductivity tensor in the outer crust of a NS is calculated in the presence of non-quantizing magnetic fields.
- The linearized Boltzmann kinetic equation is solved in relexation time approximation to obtain formulae for three components of the conductivity tensor.
- The temperature-density range where the ionic component of the crust is in liquid state has been considered.
- The ion-ion correlation is taken into account via OCP 2-point structure factor.
- In the electron-ion scattering also the dynamical screening is included.
- The electron-ion interaction is implemented in terms of hard-thermal-loop polarization tensor of QED plasma, taken in the relevant low-frequency limit.
- The crust composition has been assumed to be temperature-independent.
- The transition from the degenerate to the non-degenerate regime has been studied.
- For the components of the conductivity tensor accurate fit formulae are obtained.

THANK YOU FOR ATTENTION!