Differential rotation and nonlinear r-modes in magnetized neutron stars

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Motivation

Rezzolla et al. (2000-2001): magnetic damping of r-modes

- r-modes are likely to be accompanied by differential drift of fluid elements
- Differential drift enhance magnetic field
- Enhancement of magnetic field removes energy from r-modes => r-mode instability might be prevented or suppressed

Does this mechanism actually work in neutron stars?

Can magnetic field suppress drift before it suppresses the instability?

Talk overview

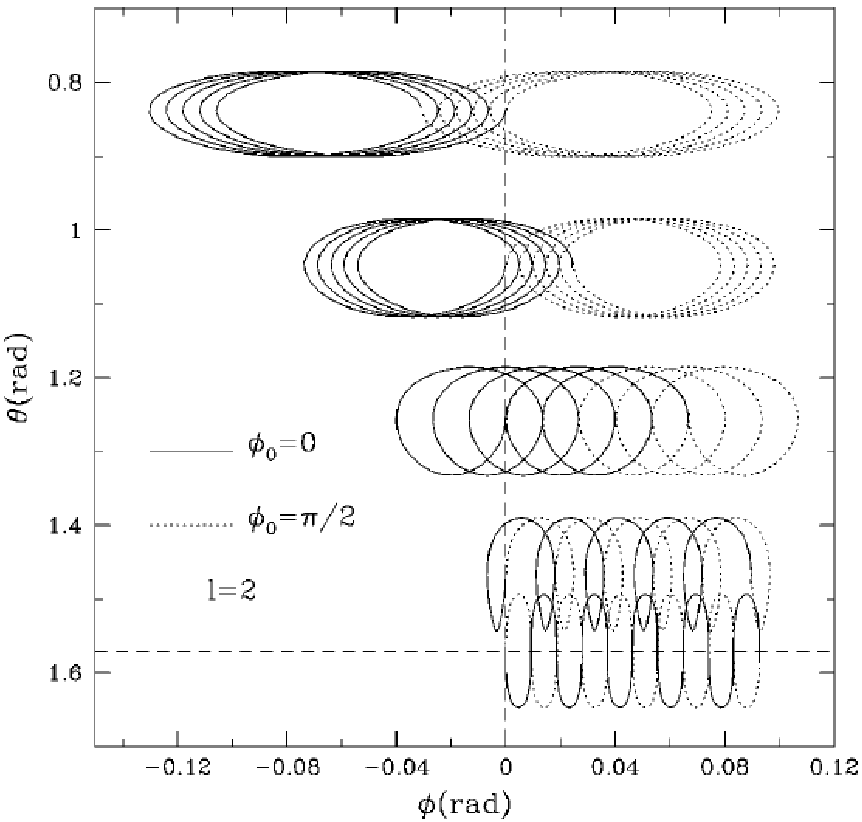
- Motivation: magnetic damping of r-modes

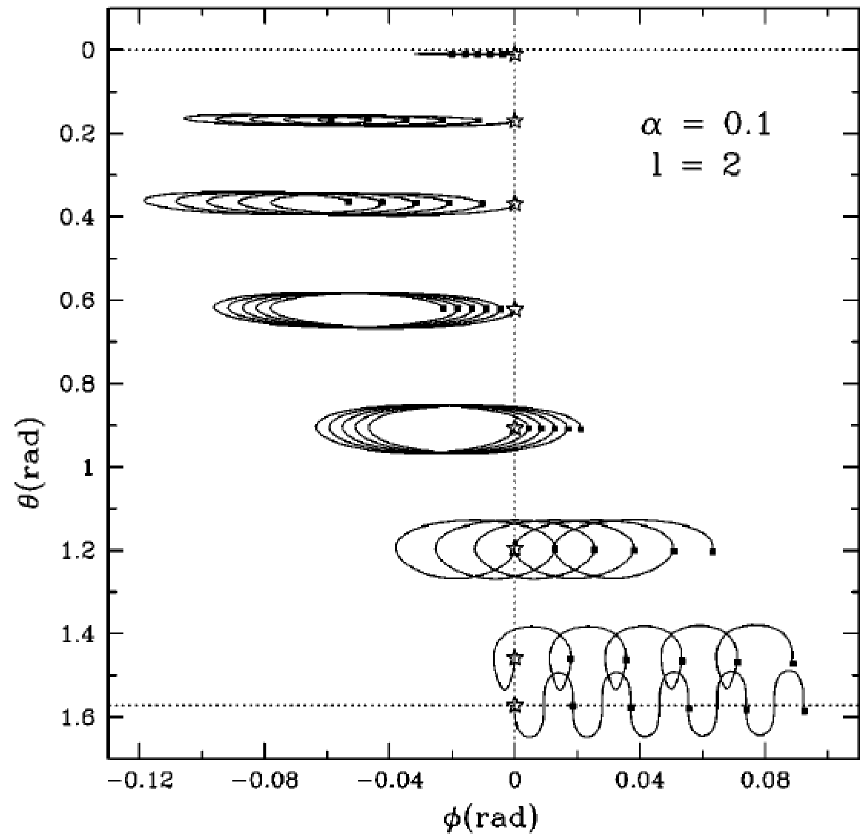
Does this mechanism actually work in neutron stars?

Yes	No
R-mode instability can generate magnetic field, but after that NS should be r-mode stable	R-mode instability put constraints to (micro)physics of NSs (shear viscosity in npe-matter is not enough)

- Introduction: Differential rotation, Stokes drift, r-mode instability and angular momentum of NS
- Second order r-modes in magnetized Newtonian NSs
- Post-Newtonian NSs: Effects of gravitational radiation-reaction force
- Summary

Rezzolla et al. (2001a):
Differential drift of fluid elements enhance magnetic field





$$\frac{\partial B}{\partial t} = [\nabla \times [\boldsymbol{v} \times \boldsymbol{B}]]$$

Magnetic field is winded up and enhanced

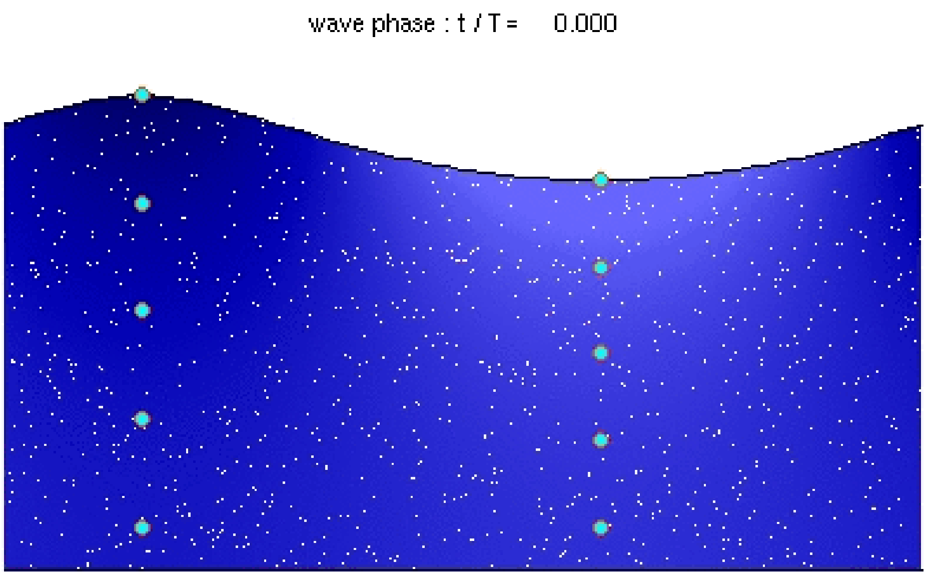
$$\boldsymbol{v} = \delta^{(1)}\boldsymbol{v}$$

Secular increase of magnetic field

$$\Delta B \propto \alpha^2 t$$

Stokes drift

G.G. Stokes (1847). "On the theory of oscillatory waves". Transactions of the Cambridge Philosophical Society ***8****: 441–455*



(from wikipedia)

Wikipedia:
For a pure wave motion in fluid dynamics, the **Stokes drift velocity** is the average velocity when following a specific fluid parcel as it travels with the fluid flow.

Main features of Stokes drift:

- Second order in amplitude
- Even if time averaged Eulerian velocity (even at every point) is (exactly) zero, macroscopic drift of fluid elements can take place
- This drift is not correction to the Eulerian velocity field

Stokes drift and Eulerian velocity profile

Second order perturbation theory

Eulerian velocity: $$v = \epsilon v_1 + \epsilon^2 v_2 + \ldots; \quad \epsilon \ll 1$$

Velocity of the fluid element, which was at coordinate x_0 at the moment t=0:

$$V(x_0, t) = v\left(x_0 + \int_0^t V \mathrm{d}t, t\right) \approx v(x_0, t) + \int_0^t V \mathrm{d}t \cdot \nabla v(x_0, t)$$

$$V = \epsilon V_1 + \epsilon^2 V_2 + \ldots$$

$$\overline{V}_1 = \overline{v}_1; \quad \overline{V}_2 = \overline{v}_2 + \boxed{\overline{\int_0^t v_1 \mathrm{d}t \cdot \nabla v_1}}$$

Stokes drift is NOT the only contribution to secular drift

Longuet-Higgins M. S., Phil. Trans. R. Soc. Long. A, **245** (1953), 535

R-mode instability and angular momentum

Excitation energy $E_{\rm ex} = E - E_{\rm rot}(J) > 0; \quad \delta E_{\rm rot} = \Omega \delta J$

Gravitational radiation: $-\frac{\omega}{m} \dot{j}^{\rm GW} = \dot{E}^{\rm GW} < 0$

$$\dot{E}^{\rm GW}_{\rm ex} = \dot{E}^{\rm GW} - \dot{E}^{\rm GW}_{\rm rot} = -\left(\Omega + \frac{\omega}{m}\right) \dot{j}^{\rm GW}$$

Enhancement of excitation energy $\dot{E}^{\rm GW}_{\rm ex} > 0$ => instability

CFS instability criteria:
[Friedman&Schutz (1978a,b)]

$$\omega + m\Omega > 0; \ \omega/m < 0$$

Rotation energy acts as energy source for CFS instability ⇒ Physical angular momentum should decrease $\delta J < 0$

R-mode instability and (differential) second order rotation

Rotation energy acts as energy source for instability ⇒ Physical angular momentum should decrease $\delta J < 0$

$$\boldsymbol{J} = \int \rho[\boldsymbol{r} \times \boldsymbol{v}]\mathrm{d}V \qquad \delta\boldsymbol{J} = \boldsymbol{J} - \boldsymbol{J}_0$$

$$\delta\rho = O(\Omega^2) = o(\Omega) \Rightarrow \delta^{(i)}\boldsymbol{J} = \int \rho[\boldsymbol{r} \times \delta^{(i)}\boldsymbol{v}]\mathrm{d}V$$

$$\delta^{(1)}v \propto \exp(im\phi) \Rightarrow \delta^{(1)}\boldsymbol{J} = 0; \quad \delta^{(2)}\boldsymbol{J} = \int \rho[\boldsymbol{r} \times \delta^{(2)}_{sym}\boldsymbol{v}]\mathrm{d}V$$

$\delta^{(2)}_{sym}\boldsymbol{v} \neq 0$ is essential for instability, but should ***drift*** be differential?
(fixed δ*J* does not fix velocity perturbation profile and thus drift profile)

Is differential drift essential for Newtonian r-modes (no GW radiation)?

Newtonian r-modes: the second order in oscillation amplitude

[Sa, Phys. Rev. D **69** (2004), 084001]

$$\begin{aligned}
\partial_t \delta^{(1)} v_i + \delta^{(1)} v^k \nabla_k v_i + v^k \nabla_k \delta^{(1)} v_i &= -\nabla_i \delta^{(1)} U, \\
\partial_t \delta^{(1)} \rho + v^i \nabla_i \delta^{(1)} \rho + \nabla_i (\rho \delta^{(1)} v^i) &= 0, \\
\triangle \delta^{(1)} \Phi &= 4\pi G \delta^{(1)} \rho,
\end{aligned}$$

$$\delta^{(i)} U = \delta^{(i)} \left(p/\rho \right) + \delta^{(i)} \Phi$$

$$\begin{aligned}
\partial_t \delta^{(2)} v_i &+ \delta^{(2)} v^k \nabla_k v_i + v^k \nabla_k \delta^{(2)} v_i + \boxed{\delta^{(1)} v^k \nabla_k \delta^{(1)} v_i} \\
&= -\nabla_i \delta^{(2)} U + \boxed{\frac{\delta^{(1)} \rho}{\rho} \nabla_i \left(\frac{\delta^{(1)} p}{p} \right)}, \\
\partial_t \delta^{(2)} \rho &+ v^i \nabla_i \delta^{(2)} \rho + \nabla_i (\rho \delta^{(2)} v^i) \\
&+ \boxed{\nabla_i (\delta^{(1)} \rho \delta^{(1)} v^i)} = 0, \\
\triangle \delta^{(2)} \Phi &= 4\pi G \delta^{(2)} \rho.
\end{aligned}$$

Induced by the first order solution

Newtonian r-modes: the second order in oscillation amplitude

[Sa, Phys. Rev. D **69** (2004), 084001]

$$
\begin{aligned}
\partial_t \delta^{(2)} v_i &+ \delta^{(2)} v^k \nabla_k v_i + v^k \nabla_k \delta^{(2)} v_i + \boxed{\delta^{(1)} v^k \nabla_k \delta^{(1)} v_i} \\
&= -\nabla_i \delta^{(2)} U + \boxed{\frac{\delta^{(1)}\rho}{\rho} \nabla_i \left(\frac{\delta^{(1)} p}{p} \right)}, \\
\partial_t \delta^{(2)} \rho &+ v^i \nabla_i \delta^{(2)} \rho + \nabla_i (\rho \delta^{(2)} v^i) \\
&+ \boxed{\nabla_i (\delta^{(1)} \rho \delta^{(1)} v^i)} = 0, \\
\triangle \delta^{(2)} \Phi &= 4\pi G \delta^{(2)} \rho.
\end{aligned}
$$

<u>Crucial properties</u>

- General second order solution = partial solution + general solution of homogeneous equations (i.e., it is not unique)
- Homogeneous equations are the same as equation for linear perturbations in nonoscillating star

Newtonian r-modes: drift velocity

[Chugunov, MNRAS **451,** 2772–2779 (2015)]

$$\left(v^{(\mathrm{d})}\right)^{\phi} = \alpha^2 \Omega f(\varpi) \qquad \varpi = r\sin(\theta)$$

Drift velocity is a solution of linearized equations:
arbitrary (differential) rotation, stratified on cylinders.
Cylindrical layers are decoupled and moves independently

It is likely to be general property of oscillation modes

Longuet-Higgins M. S., Phil. Trans. R. Soc. Long. A, **245** (1953), 535:

Averaged acceleration of fluid elements is (at least) 3-th order in amplitude

Moore D., Geophysical and Astrophysical Fluid Dynamics, **1** (1970), 237

Abstract—Second order effects due to the presence of a first order free oscillation at a single frequency in a variable depth rotating ocean are examined. It is found that the second order Lagrangian mean velocity (mass transport velocity) satisfies the linearized equations for unforced steady geostrophic motion. This implies that if the ocean basin is laterally bounded and contains no closed geostrophic contours, the second order Lagrangian mean velocity vanishes everywhere.

Newtonian NS: Drift and oscillations are decoupled

[Chugunov, MNRAS **451,** 2772–2779 (2015)]

Drift and oscillations are independent degrees of freedom:
General second order r-mode solution: superposition of
(a) oscillating solution with *vanishing drift*
(b) drift
Secular evolution of magnetic field is coupled only with drift

$$B = 0$$

Oscillating solution with vanishing drift exists

Drift: Arbitrary stationary motion of cylindrical layers

$$B \neq 0$$

Oscillating solution with vanishing drift is unaffected by *B*, if $B \ll 10^{16}$ G

Drift & magnetic field evolves as in nonoscillating star (Alfven waves + uniform rotation).

R-mode energy is conserved. Magnetic damping is absent

Effect of gravitational radiation-reaction force

In absence of magnetic field: exponentially increasing of mode amplitude and displacement of fluid elements [Friedman et al., 2016]

$$\alpha = \alpha_0 \exp(\beta t)$$

R-mode energy density: $\dot{\epsilon}_{\mathrm{GW}} \approx 2\beta\epsilon \sim \rho\alpha^2\beta\,\Omega^2 R^2$

Drift:

$$\langle\dot{\phi}(t)\rangle \approx -\frac{3}{2}\alpha^2(t)\Omega\left[\frac{\varpi^2}{4} + \Upsilon(\varpi)\right]; \quad \varpi = r\sin(\theta)$$

Drift is cylindrically stratified, but differential.
As far as cylindrical layers are decoupled it is not surprising

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Magnetic field: Couples cylindrical layers at the Alfven timescale

$$\tau_{\mathrm{A}} = R/\sqrt{B^2/(4\pi\rho)}$$

Effect of radiation-reaction force: small magnetic field

$$\beta\tau_{\rm A} \gg 1$$

Drift is unaffected by magnetic field.
Magnetic field enhanced at the r-mode evolution timescale β

$$B(t) \sim B_0\,\Delta\phi \sim B_0\alpha_0^2\frac{\Omega}{\beta}\exp(2\beta t);\;\; \dot{B} \sim \alpha^2\Omega B_0$$

$$\dot{\epsilon}_{\rm mag} \approx \frac{B}{2\pi}\dot{B} \sim \alpha^4\frac{\Omega^2}{\beta}B_0^2$$

$$\dot{\epsilon}_{\rm GW} \sim \rho\alpha^2\beta\,\Omega^2 R^2$$

$$\dot{\epsilon}_{\rm mag} \sim \alpha^2(\beta\tau_A)^{-2}\dot{\epsilon}_{\rm GW} \ll \dot{\epsilon}_{\rm GW}$$

Magnetic damping is negligible for small magnetic field

Effect of radiation-reaction force: large magnetic field

$$\beta\tau_{\mathrm{A}} \ll 1$$

Exponential grow of differential rotation in nonmagnetized NS is associated with axisymmetric part of gravitational radiation-reaction force [Friedman et al., PRD 93 (2016), 024023]. It can be estimated as

$$F_{\mathrm{GR}}^{\mathrm{sim}} = \alpha^2\rho\delta^{(2)} f_{\mathrm{GR}} \sim \rho\alpha^2\beta\Omega R$$

Drift generates magnetic field

$$\boldsymbol{\Delta B} = \boldsymbol{B} - \boldsymbol{B}_0$$

Perturbation of the axisymmetric magnetic force can be estimated as

$$F_{\mathrm{mag}}^{\mathrm{sim}} \sim \Delta B B_0/(4\pi R)$$

Magnetic force can modify drift profile if $\quad F_{\mathrm{GR}}^{\mathrm{sim}} \sim F_{\mathrm{mag}}^{\mathrm{sim}}$

Effect of radiation-reaction force: large magnetic field

$$\beta\tau_{\rm A} \ll 1$$

Magnetic force is crucial if $F^{\rm sim}_{\rm GR} \sim F^{\rm sim}_{\rm mag}$

$$\Delta B \sim \frac{\rho\alpha^2\,\beta\Omega R^2}{B_0} \;\Rightarrow\; \dot{B} \sim \frac{\rho\alpha^2\,\beta^2\Omega R^2}{B_0} = (\beta\tau_{\rm A})^2\alpha^2\Omega B_0$$

Magnetic field redistribute angular momentum between cylindrical layers and differential drift is suppressed for a factor of $(\beta\tau_{\rm A})^2 \ll 1$

$$\dot{\epsilon}_{\rm GW} \sim \rho\alpha^2\beta\,\Omega^2 R^2$$

$$\dot{\epsilon}_{\rm mag} \approx \frac{B\dot{B}}{2\pi} \sim \alpha^2(\beta\tau_{\rm A})^2\dot{\epsilon}_{\rm GW} \ll \dot{\epsilon}_{\rm GW}$$

Magnetic damping is negligible for strong magnetic field

Effect of radiation-reaction force: saturation

$$\alpha = \alpha_{\rm sat} = {\rm const} \quad \Rightarrow \quad F_{\rm GR}^{\rm sim} = {\rm const}$$

$$F_{\rm GR}^{\rm sim} \sim F_{\rm mag}^{\rm sim} \qquad \Delta B \sim \frac{\alpha_{\rm sat}^2 \Omega \beta \rho R^2}{B_0} = {\rm const}$$

$$\Downarrow$$

$$\dot{B} = 0; \quad \epsilon_{\rm mag} = 0$$

Magnetic damping vanishes

Friedman et al. (2016): "After nonlinear saturation, we expect the growth of differential rotation and of the magnetic field to stop within a time on the order of the Alfvén time".

Summary

Newtonian NS (absence of radiation-reaction force):

- r-modes can exist with vanishing differential drift (but second order axisymmetric perturbations of Eulerian velocity are differential)
- Drift and r-mode oscillations are decoupled

Post-Newtonian NS (radiation reaction force included):

- Strong magnetic field $(\beta\tau_{\rm A} \ll 1)$ strongly suppress differential drift and magnetic damping becomes negligible

$$\dot{\epsilon}_{\rm mag} \sim \alpha^2(\beta\tau_{\rm A})^2\dot{\epsilon}_{\rm GW} \ll \dot{\epsilon}_{\rm GW}$$

- Small magnetic field $(\beta\tau_{\rm A}) \gg 1$ is not enhanced by r-modes enough strongly to suppress r-mode instability

$$\dot{\epsilon}_{\rm mag} \sim \alpha^2(\beta\tau_A)^{-2}\dot{\epsilon}_{\rm GW} \ll \dot{\epsilon}_{\rm GW}$$

- Saturated r-modes does not enhance magnetic field

Magnetic damping seems to be irrelevant for r-mode instability
R-mode instability put constraints to the (micro)physics